Quadrupolar gravitational radiation as a test-bed for $f(R)$-gravity

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The debate concerning the viability of $f(R)$-gravity as a natural extension of General Relativity could be realistically addressed by using results coming from binary pulsars like PSR 1913+16. To this end, we develop a quadrupolar approach to the gravitational radiation for a class of analytic $f(R)$-models. We show that experimental results are compatible with a consistent range of $f(R)$-models. This means that $f(R)$-gravity is not ruled out by the observations and gravitational radiation (in strong field regime) could be a test-bed for such theories.

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I. INTRODUCTION

The discovery of binary pulsars PSR 1913+16 by Hulse and Taylor in 1974 [1] opened a new testing ground for General Relativity (GR). In fact its continuous observation by Taylor and coworkers [2, 3], led to an impressively accurate tracking of the orbital motion of the binary system. Before this discovery, the only available testing ground for GR was the Solar System where the gravitational field is slowly varying and represents only a very small deformation of a flat space-time. As a consequence, Solar System tests can only prove the weak-field limit of GR. By contrast, binary systems containing compact objects as neutron stars (NS) or black holes (BH) involve space-time domains where the gravitational field is strong. Indeed, the gravitational field on the surface (*i.e.* $\simeq 2\,GM/c^2R$) of a NS is of order 0.4, which is close to the one of a BH $\simeq 2\,GM/c^2R = 1$) and much larger than the gravitational field on surfaces of Solar System bodies: $\simeq (2\,GM/c^2R)_\odot \sim 10^{-6}$, $(2\,GM/c^2R)_\oplus \sim 10^{-9}$. In addition, the high stability of *pulsar clocks* has made it possible to monitor the dynamics of its orbital motion down to a precision allowing one to measure the small ($\sim (v/c)^5$) orbital effects linked to the propagation of the gravitational field at the velocity of light between the pulsar and its companion. The recent discoveries of the double binary pulsars [4, 5] has renewed the interest in the use of binary pulsars as extremely relevant test-beds of gravity theories. This means that it is worth reconsidering in detail, *i.e.* at its foundation, the problem of motion also in relation to the problem of generation and detection of gravitational waves (GWs). In other words, the motion of sources could give further signatures to GWs and then it has to be carefully reconsidered.

The achieved sensitivity levels and theoretical developments are leading toward a general picture of GW phenomena that was not possible in the previous pioneering era. Experimentally, several GW ground-based laser interferometer detectors have been built in the United States (LIGO) [6], Europe (VIRGO and GEO) [7, 8] and Japan (TAMA) [9], and are now taking data at designed sensitivities. A laser-interferometer space antenna (LISA) [10] might fly within the next decade. As results, we can hope that the next decade will witness the direct detection of gravitational waves opening the fields of GW astronomy and cosmology. Theoretical studies have been developed in parallel to the experimental activity. In particular, mechanisms for the production of GWs, both in astrophysics and in cosmology. Templates on binary inspiral (see *e.g.* [11–14]) and robust search algorithms have been developed for GWs sources [15]. Furthermore conceptual and technical problems, related to the production of GWs by self-gravitating systems (such as coalescing binaries) have not been fully solved. This status of art suggests to reconsider the problems of motion and generation of GWs also with respect to alternative theories of gravity which seem realistic approaches to face several problems in astrophysics and cosmology. In particular $f(R)$-gravity seem a viable semi-classical scheme to overcome shortcomings related to infrared and ultraviolet behaviors of the gravitational field [16]. These theories are based on corrections and enlargements of the Einstein GR. Besides fundamental physics motivations, they have acquired interest in cosmology due to the fact that they "naturally" exhibit inflationary behaviors able to overcome the shortcomings of Standard Cosmological Model (based on GR). The related cosmological models seem realistic and, several times, capable of matching with the observations [17–19, 21]. From a genuine astrophysical viewpoint, these Extended Theories of Gravity (ETGs) [20] do not urgently require to find out candidates for dark energy and dark matter at fundamental level (till now they have not been detected!). The approach is very conservative taking into account only the "actually observed" ingredients (*i.e.* gravity, radiation and baryonic matter); it is in full agreement with the early spirit of GR which could not act in the same way at all scales (see [20] for a comprehensive review). In fact, GR has been successfully probed in the weak-field limit (*e.g.* Solar System experiments) and also in this case there is room for alternative theories of gravity which are not at all ruled out, as discussed in several recent studies [22–24]. In particular, it is possible to show that several $f(R)$-models could satisfy both cosmological and Solar System tests [25, 26], could be constrained as the scalar-tensor theories and could give rise to new effects

capable of explaining anomalies also at local scales (see for example [27] and references therein).

In this paper we study the quadrupolar gravitational radiation in $f(R)$-gravity using the "linearized theory". It consists in expanding the field equations around the flat Minkowski metric. The field equations then reduce to linear wave equations from which radiation can be calculated. GR predicts radiation that, at the lowest order, is proportional to the third derivative of the quadrupole momentum of the mass-energy distribution. It is a consequence of conservation equations that the first derivative of the monopole momentum and the second derivative of the dipole momentum are zero. This means that the gravitational radiation is first seen at the quadrupole term. The dipole effects depends on the difference of the self-gravitational binding energy per unit mass for two bodies and it is thus dependent also on the internal structures of the objects. When the objects are in circular orbits, the time variation of the scalar field at each object, due to the motion of the other, is zero and the dipole contributions consequently drop out. Under these circumstances the dominant surviving terms are of quadrupole order. In $f(R)$-gravity the situation is different due to the presence of further degrees of freedom of the gravitational field [29–31]. However, GR has to be fully recovered as soon as $f(R) \to R$. This "compatibility" with GR could be a test-bed for these ETGs. Here we develop expressions for quadrupole gravitational radiation in $f(R)$-gravity using the weak field technique and apply these results, to binary systems as, for example, the well known PSR 1913+16. In this way, it is straightforward to compare the GR-predictions with those of ETGs. The outline of the paper is the following. In Sec. II, we briefly introduce the weak field limit and field equations of $f(R)$-gravity. Secs. III and IV are devoted to the calculation of the conservation laws. Finally the application to PSR 1913+16 is developed in Sec. V. Conclusions are drawn in Sec. VI.

II. FIELD EQUATIONS AND POST-MINCOWSKIAN LIMIT OF $f(R)$-GRAVITY

The post-Minkowskian limit of any theory of gravity arises when the regime of small field is considered without any prescription on the propagation of the field. This case has to be clearly distinguished with respect to the Newtonian limit which, differently, requires both the small velocity and the weak field approximations. Often, in literature, such a distinction is not clearly remarked and several cases of pathological analysis can be accounted. The post-Minkowskian limit of GR gives rise to massless gravitational waves. An analogous study can be pursued considering, instead of the Hilbert-Einstein Lagrangian linear in the Ricci scalar R, a general function $f(R)$ [32]. The only assumption that we are going to do is that $f(R)$ is an analytic function. The gravitational action is then

$$\mathcal{A} = \int d^4x\sqrt{-g}\Big[f(R) + \mathcal{X}\mathcal{L}_m\Big]\,, \tag{1}$$

where $\mathcal{X} = \dfrac{16\pi G}{c^4}$ is the coupling, $\mathcal{L}_m$ is the standard matter Lagrangian and g is the determinant of the metric[1]. The field equations, in metric formalism, read[2]

$$f'(R)R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\Box_g f'(R) = \frac{\mathcal{X}}{2}T_{\mu\nu}\,, \tag{2}$$

$$3\Box f'(R) + f'(R)R - 2f(R) \,=\, \frac{\mathcal{X}}{2}T\,, \tag{3}$$

with $T_{\mu\nu} = \dfrac{-2}{\sqrt{-g}}\dfrac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ the energy momentum tensor of matter (T is the trace), $f'(R) = \dfrac{df(R)}{dR}$ and $\Box_g = {}_{;\sigma}{}^{;\sigma}$. We adopt a $(+,-,-,-)$ signature, while the conventions for Ricci's tensor is $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$ and $R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} + ...$ for the Riemann tensor, where

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})\,, \tag{4}$$

are the Christoffel symbols of the $g_{\mu\nu}$ metric. Actually, in order to perform a post-Minkowskian limit of field equations, one has to perturb Eqs. (2) on the Minkowski background $\eta_{\mu\nu}$. In such a case the invariant metric element becomes

$$ds^2 = g_{\sigma\tau}dx^\sigma dx^\tau = (\eta_{\sigma\tau} + h_{\sigma\tau})dx^\sigma dx^\tau\,, \tag{5}$$

with $h_{\mu\nu}$ small ($\mathcal{O}(h)^2 \ll 1$). We assume that the $f(R)$-Lagrangian is analytic (*i.e.* Taylor expandable) in term of the Ricci scalar, which means that[3]

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R-R_0)^n \simeq f_0 + f'_0R + \frac{1}{2}f''_0R^2 + ...\,. \tag{6}$$

The flat-Minkowski background is recovered for $R = R_0 \simeq 0$.

Field equations (2), at the first order of approximation in term of the perturbation [34], become:

[1] Here we indicates with "," partial derivative and with " ;" covariant derivative with regard to $g_{\mu\nu}$; all Greek indices run from $0, ..., 3$ and Latin indices run from $1, ..., 3$; g is the determinant.

[2] All considerations are developed here in metric formalism. From now on we assume physical units $G = c = 1$.

[3] for convenience we will use for the following calculations, f instead of $f(R)$

$$f_0'\left[R^{(1)}_{\mu\nu}-\frac{R^{(1)}}{2}\eta_{\mu\nu}\right]-f_0''\left[R^{(1)}_{,\mu\nu}-\eta_{\mu\nu}\Box R^{(1)}\right]=\frac{\mathcal{X}}{2}T^{(0)}_{\mu\nu} \quad (7)$$

where $f_0'=\frac{df}{dR}\Big|_{R=0}$, $f_0''=\frac{d^2f}{dR^2}\Big|_{R=0}$ and $\Box={}_{,\sigma}{}^{,\sigma}$ that is now the standard d'Alembert operator of flat spacetime. From the zero-order of Eqs.(2), one gets $f(0)=0$, while $T_{\mu\nu}$ is fixed at zero-order in Eq.(7) since, in this perturbation scheme, the first order on Minkowski space has to be connected with the zero order of the standard matter energy momentum tensor[4]. The explicit expressions of the Ricci tensor and scalar, at the first order in the metric perturbation, read

$$\begin{cases} R^{(1)}_{\mu\nu}=h^{\sigma}_{(\mu,\nu)\sigma}-\frac{1}{2}\Box h_{\mu\nu}-\frac{1}{2}h_{,\mu\nu} \\ \\ R^{(1)}=h_{\sigma\tau}{}^{,\sigma\tau}-\Box h \end{cases} \quad (8)$$

with $h=h^{\sigma}{}_{\sigma}$. Eqs. (7) can be written in a more suitable form by introducing the constant $\xi=-\frac{f_0''}{f_0'}$, that is

$$h^{\sigma}_{(\mu,\nu)\sigma}-\frac{1}{2}\Box h_{\mu\nu}-\frac{1}{2}h_{,\mu\nu}-\frac{1}{2}(h_{\sigma\tau}{}^{,\sigma\tau}-\Box h)\eta_{\mu\nu}$$
$$+\xi(\partial^2_{\mu\nu}-\eta_{\mu\nu}\Box)(h_{\sigma\tau}{}^{,\sigma\tau}-\Box h)=\frac{\mathcal{X}}{2f_0'}T^{(0)}_{\mu\nu}\,. \quad (9)$$

By choosing the transformation $\tilde{h}_{\mu\nu}=h_{\mu\nu}-\frac{h}{2}\eta_{\mu\nu}$ and the gauge condition $\tilde{h}^{\mu\nu}{}_{,\mu}=0$, one obtains that field equations and the trace equation, respectively, read [5]

$$\begin{cases} \Box\tilde{h}_{\mu\nu}+\xi(\eta_{\mu\nu}\Box-\partial^2_{\mu\nu})\Box\tilde{h}=-\frac{\mathcal{X}}{f_0'}T^{(0)}_{\mu\nu} \\ \\ \Box\tilde{h}+3\xi\Box^2\tilde{h}=-\frac{\mathcal{X}}{f_0'}T^{(0)} \end{cases} \quad (10)$$

It is worth noticing that solving the previous system of equations, we find wavelike solutions with massless and massive contributions [29, 31, 32]. The presence of the massive term is a feature emerging from the higher-order terms in $f(R)$-gravity. Specifically, it is related to the fact that $f_0''\neq 0$, which is null in GR where $f(R)=R$. This means that massless states are a particular case among the gravitational theories that present also massive ones. A similar situation emerges also in the Newtonian limit: the Newton potential is recovered only as the weak field limit of GR. In general, Yukawa-like corrections, and then characteristic interaction lengths, are present [33]. The effective mass is $m^2=(3\xi)^{-1}=-\frac{f_0'}{3f_0''}$ and then f_0'' has to be negative in order to have physically defined states. It is easy to see that massive modes are directly related to the non-trivial structure of the trace equation Eq.(3). In GR, the Ricci scalar is univocally fixed being $R=0$ in vacuum and $R\propto\rho$ in presence of matter, where ρ is the matter-energy density [29, 31, 32]. The task is now to evaluate the related energy-momentum tensors.

III. ENERGY-MOMENTUM TENSORS

Let us assume that the source $T_{\mu\nu}$ is localized in a finite region. Outside this region $T_{\mu\nu}=0$.

Then, as a consequence of Eqs.(8) and gauge condition, we have

$$R^{(1)}_{\mu\nu}=\Box h_{\mu\nu}=0\,, \quad (11)$$

outside the region. There are several ways to define the energy-momentum tensor of the gravitational field. One is to consider $R_{\mu\nu}$ on the left-hand side of Eq.(2) consisting of a series of correction terms in $R^{(N)}_{\mu\nu}$. In the development of Eq.(7), $R^{(1)}_{\mu\nu}$ is on the left-hand side. The remaining higher order terms, which so far have been ignored, could be brought to the right-hand side. If the source region gives rise to a flux of energy in the form of GWs, it must be represented by these higher order terms. This is the geometric approach [35]. The other approach is to use the standard field theoretical methods. The geometric and the field theoretical approaches are complementary. Some aspects of GWs physics can be better understood from the former approach, some from the latter, and to study GWs from both vantage points results in a deeper overall understanding. We use the latter approach to calculate the stress-energy tensor of the gravitational field. So one can extending the formalism to more general theories and obtain this quantity by varying the gravitational Lagrangian. In GR, this quantity is a pseudo-tensor and is typically referred to as the Landau-Lifshitz energy-momentum tensor [36].

In the case of $f(R)$-gravity, we have

$$\delta\int d^4x\sqrt{-g}f(R)=\delta\int d^4x\mathcal{L}(g_{\mu\nu},g_{\mu\nu,\rho},g_{\mu\nu,\rho\sigma})\approx$$
$$\int d^4x\left(\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma}}-\partial_\lambda\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda}}+\partial^2_{\lambda\xi}\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}}\right)\delta g_{\rho\sigma}=$$
$$\doteq\int d^4x\sqrt{-g}H^{\rho\sigma}\delta g_{\rho\sigma}=0\,. \quad (12)$$

The Euler-Lagrange equations are then

[4] This formalism descends from the theoretical setting of Newtonian mechanics which requires the appropriate scheme of approximation when obtained from a more general relativistic theory. This scheme coincides with a gravity theory analyzed at the first order of perturbation in the curved spacetime metric.

[5] The gauge transformation is $h'_{\mu\nu}=h_{\mu\nu}-\zeta_{\mu,\nu}-\zeta_{\nu,\mu}$ when we perform a coordinate transformation as $x'^{\mu}=x^{\mu}+\zeta^{\mu}$ with $O(\zeta^2)\ll 1$. To obtain the gauge and the validity of the field equations for both perturbation $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$, the ζ_μ have to satisfy the harmonic condition $\Box\zeta^{\mu}=0$.

$$\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma}}-\partial_\lambda\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda}}+\partial^2_{\lambda\xi}\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}}=0, \tag{13}$$

which coincide with the field Eqs. (2) in vacuum. Actually, even in the case of more general theories, it is possible to define an energy-momentum tensor that turns out to be defined as follows:

$$\begin{aligned} t^\lambda_\alpha &= \frac{1}{\sqrt{-g}}\left[\left(\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda}}-\partial_\xi\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}}\right)g_{\rho\sigma,\alpha}+\right.\\ &\left.+\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}-\delta^\lambda_\alpha\mathcal{L}\right]. \end{aligned} \tag{14}$$

This quantity, together with the energy-momentum tensor of matter $T_{\mu\nu}$, satisfies a conservation law as required by the Bianchi identities. In fact, in presence of matter, one has $H_{\mu\nu}=\frac{\chi}{2}T_{\mu\nu}$, and then

$$\begin{aligned} (\sqrt{-g}t^\lambda_\alpha)_{,\lambda} &= -\sqrt{-g}H^{\rho\sigma}g_{\rho\sigma,\alpha}=\\ &=-\frac{\mathcal{X}}{2}\sqrt{-g}T^{\rho\sigma}g_{\rho\sigma,\alpha}=-\mathcal{X}(\sqrt{-g}T^\lambda_\alpha)_{,\lambda}\,, \end{aligned} \tag{15}$$

and, as a consequence,

$$[\sqrt{-g}(t^\lambda_\alpha+\mathcal{X}T^\lambda_\alpha)]_{,\lambda}=0\,, \tag{16}$$

that is the conservation law given by the Bianchi identities. We can now write the expression of the energy-momentum tensor t^λ_α in term of the gravity action $f(R)$ and its derivatives:

$$\begin{aligned} t^\lambda_\alpha &= f'\left\{\left[\frac{\partial R}{\partial g_{\rho\sigma,\lambda}}-\frac{1}{\sqrt{-g}}\partial_\xi\left(\sqrt{-g}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}\right)\right]g_{\rho\sigma,\alpha}\right.\\ &\left.+\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}\right\}-f''R_{,\xi}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\alpha}-\delta^\lambda_\alpha\ f\,, \end{aligned} \tag{17}$$

It is worth noticing that t^λ_α is a non-covariant quantity in GR while its generalization, in fourth order gravity, turns out to satisfy the covariance prescription of standard tensors (see also [37]). On the other hand, such an expression reduces to the Landau-Lifshitz energy-momentum tensor of GR as soon as $f(R)=R$, that is

$$t^\lambda_{\alpha|_{\rm GR}}=\frac{1}{\sqrt{-g}}\left(\frac{\partial\mathcal{L}_{\rm GR}}{\partial g_{\rho\sigma,\lambda}}g_{\rho\sigma,\alpha}-\delta^\lambda_\alpha\mathcal{L}_{\rm GR}\right), \tag{18}$$

where the GR Lagrangian has been considered in its effective form, *i.e.* the symmetric part of the Ricci tensor, which effectively leads to the equations of motion, that is

$$\mathcal{L}_{\rm GR}=\sqrt{-g}g^{\mu\nu}(\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}-\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho})\,. \tag{19}$$

It is important to stress that the definition of the energy-momentum tensor in GR and in $f(R)$-gravity are different. This discrepancy is due to the presence, in the second case, of higher than second order differential terms that cannot be discarded by means of a boundary integration as it is done in GR. We have noticed that the effective Lagrangian of GR turns out to be the symmetric part of the Ricci scalar since the second order terms, present in the definition of R , can be removed by means of integration by parts.

On the other hand, an analytic $f(R)$-Lagrangian can be recast, at linear order, as $f\sim f'_0R+\mathcal{F}(R)$, where the function $\mathcal{F}$ satisfies the condition: $\lim_{R\to 0}\mathcal{F}\to R^2$. As a consequence, one can rewrite the explicit expression of t^λ_α as:

$$\begin{aligned} t^\lambda_\alpha &= f'_0t^\lambda_{\alpha|_{\rm GR}}+\\ &+\mathcal{F}'\left\{\left[\frac{\partial R}{\partial g_{\rho\sigma,\lambda}}-\frac{1}{\sqrt{-g}}\partial_\xi\left(\sqrt{-g}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}\right)\right]g_{\rho\sigma,\alpha}\right.\\ &\left.+\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}\right\}-\mathcal{F}''R_{,\xi}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\alpha}-\delta^\lambda_\alpha\mathcal{F}. \end{aligned} \tag{20}$$

The general expression of the Ricci scalar, obtained by splitting its linear (R^*) and quadratic ($\bar{R}$) parts once a perturbed metric (5) is considered, is

$$\begin{aligned} R &= g^{\mu\nu}(\Gamma^\rho_{\mu\nu,\rho}-\Gamma^\rho_{\mu\rho,\nu})+g^{\mu\nu}(\Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{\mu\nu}-\Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\nu\sigma})=\\ &=R^*+\bar{R}\,, \end{aligned} \tag{21}$$

(notice that $\mathcal{L}_{\rm GR}=-\sqrt{-g}\bar{R}$). In the case of GR $t^\lambda_{\alpha|_{\rm GR}}$, the Landau-Lifshitz tensor presents a first non-vanishing term at order h^2. A similar result can be obtained in the case of $f(R)$-gravity. In fact, taking into account Eq.(20), one obtains that, at the lower order, t^λ_α reads:

$$\begin{aligned} t^\lambda_\alpha &\sim t^\lambda_{\alpha|h^2}=f'_0t^\lambda_{\alpha|_{\rm GR}}+\\ &+f''_0R^*\left[\left(-\partial_\xi\frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}}\right)g_{\rho\sigma,\alpha}+\frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}\right]-\\ &+f''_0R^*_{,\xi}\frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\alpha}-\frac{1}{2}f''_0\delta^\lambda_\alpha R^{*2}=\\ &= f'_0t^\lambda_{\alpha|_{\rm GR}}+f''_0\left[R^*\left(\frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}-\frac{1}{2}R^*\delta^\lambda_\alpha\right)-\right.\\ &\left.+\partial_\xi\left(R^*\frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}}\right)g_{\rho\sigma,\alpha}\right]. \end{aligned} \tag{22}$$

Considering the perturbed metric (5), we have $R^*\sim R^{(1)}$, where $R^{(1)}$ is defined as in (8). In terms of h and η, we get

$$\begin{cases} \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}}\sim\frac{\partial R^{(1)}}{\partial h_{\rho\sigma,\lambda\xi}}=\eta^{\rho\lambda}\eta^{\sigma\xi}-\eta^{\lambda\xi}\eta^{\rho\sigma}\\ \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}\sim h^{\lambda\xi}{}_{,\xi\alpha}-h^{,\lambda}{}_\alpha \end{cases}. \tag{23}$$

Clearly, the first significant term in Eq. (22) is of second order in the perturbation expansion. We can now write the expression of the energy-momentum tensor explicitly in term of the perturbation h; it is

$$\begin{aligned} t^\lambda_\alpha \ \sim \ & f'_0 t^\lambda_{\alpha|_{\rm GR}} + f''_0\{(h^{\rho\sigma}{}_{,\rho\sigma} - \Box h)\Big[h^{\lambda\xi}{}_{,\xi\alpha} - h^{,\lambda}{}_{\alpha} - \\ & + \frac{1}{2}\delta^\lambda_\alpha(h^{\rho\sigma}{}_{,\rho\sigma} - \Box h)\Big] - h^{\rho\sigma}{}_{,\rho\sigma\xi} h^{\lambda\xi}{}_{,\alpha} + \\ & + h^{\rho\sigma}{}_{,\rho\sigma}{}^{\lambda} h_{,\alpha} + h^{\lambda\xi}{}_{,\alpha}\Box h_{,\xi} - \Box h^{,\lambda} h_{,\alpha}\} \,. \end{aligned} \tag{24}$$

Considering the tilded perturbation metric $\tilde{h}_{\mu\nu}$, the more compact form

$$\begin{aligned} t^\lambda_{\alpha|_f} \ = \ & \Big[\frac{1}{4}\tilde{h}^{,\lambda}{}_{\alpha}\Box\tilde{h} - \frac{1}{4}\tilde{h}_{,\alpha}\Box\tilde{h}^{,\lambda} - \frac{1}{2}\tilde{h}^{\lambda}{}_{\sigma,\alpha}\Box\tilde{h}^{,\sigma} - \\ & + \frac{1}{8}(\Box\tilde{h})^2\delta^\lambda_\alpha\Big] \,, \end{aligned}$$

is achieved.

As matter of facts, the energy-momentum tensor of the gravitational field, which expresses the energy transport during the propagation, has a natural generalization in the case of $f(R)$-gravity. We have adopted here the Landau-Lifshitz definition but other approaches can be taken into account [38]. The general definition of t^λ_α, obtained above, consists of a sum of a GR contribution plus a term coming from $f(R)$-gravity:

$$t^\lambda_\alpha = f'_0 t^\lambda_{\alpha|_{\rm GR}} + f''_0 t^\lambda_{\alpha|_f} \,. \tag{25}$$

However, as soon as $f(R) = R$, we obtains $t^\lambda_\alpha = t^\lambda_{\alpha|_{\rm GR}}$. As a final remark, it is worth noticing that massive modes of gravitational field come out from $t^\lambda_{\alpha|_f}$ since $\Box\tilde{h}$ can be considered an effective scalar field moving in a potential: t^λ_α, in this case, represents a transport tensor.

The expression for the gravitational tensor t^λ_α can be simplified by doing approximations valid far from the source region. Far from the source $h_{\mu\nu}$ will be, functions of a single scalar variable t'

$$t' = t - r \,, \tag{26}$$

where

$$r^2 = x_i x^i \,. \tag{27}$$

Such a scalar can be constructed from the vector x^μ by forming

$$t' = k_\lambda x^\lambda \,, \tag{28}$$

with

$$k_0 \equiv -k^0 \equiv 1 \,, \qquad k_i \equiv -\hat{x}_i \,, \tag{29}$$

$$\hat{x}_i \equiv \frac{x^i}{r} \,. \tag{30}$$

In the far field k_λ can be considered as a constant vector, over a small region. That is, $h_{\mu\nu}$ will be almost plane. Any $\frac{1}{r}$ variation or change of the unit vector $\hat{x}$ over points in the region can be made arbitrarily small by choosing a region far from the source [35].

The functional dependency of solutions will be on the $t' = t - r$. This fact can be done by expressing all partials of $h_{\mu\nu}$ as

$$h_{\mu\nu,\sigma} = \frac{\partial t'}{\partial x^\sigma}\frac{dh_{\mu\nu}}{dt'} = k_\lambda\delta^\lambda_\sigma\dot{h}_{\mu\nu} = k_\sigma\dot{h}_{\mu\nu} \,, \tag{31}$$

where

$$h_{\mu\nu} = h_{\mu\nu}\left(k_\lambda x^\lambda\right) = h_{\mu\nu}(t') \,, \tag{32}$$

here the dot indicate the derivative with respect to the time and $\frac{\partial x^\lambda}{\partial x^\sigma} = \delta^\lambda_\sigma$. Since $T_{\mu\nu} = 0$ outside the source region,

$$\Box h_{\mu\nu} = 0 \,, \tag{33}$$

in the far field [29, 31]. If Eq.(31) is used in the first of these, we find

$$\Box h_{\mu\nu} = h_{\mu\nu,\rho,}{}^{\rho} = \left(k_\rho\dot{h}_{\mu\nu}\right){}^{,\rho} = k_\rho k^\rho\ddot{h}_{\mu\nu} \,, \tag{34}$$

implying that

$$k_\rho k^\rho = 0 \,. \tag{35}$$

Therefore, from Eq.(24), the energy-momentum tensor associated with the tensor part of the gravitational field is

$$\begin{aligned} t^\lambda_\alpha \ = \ & f'_0\left(k^\lambda k_\alpha\dot{h}^{\rho\sigma}\dot{h}_{\rho\sigma}\right) + f''_0\Big(k_\rho k_\sigma\ddot{h}^{\rho\sigma}k_\xi k_\alpha\ddot{h}^{\lambda\xi} - \\ & + k_\rho k_\sigma\ddot{h}^{\rho\sigma}k^\lambda k_\alpha\ddot{h} - \frac{1}{2}k_\rho k_\sigma\ddot{h}^{\rho\sigma}\delta^\lambda_\alpha k_\rho k_\sigma\ddot{h}^{\rho\sigma} + \\ & + \frac{1}{2}k_\rho k_\sigma\ddot{h}^{\rho\sigma}\delta^\lambda_\alpha\Box h - k_\xi k_\alpha\ddot{h}^{\lambda\xi}\Box h + k^\lambda k_\alpha\ddot{h}\Box h + \\ & + \frac{1}{2}\delta^\lambda_\alpha k_\rho k_\sigma\ddot{h}^{\rho\sigma}\Box h - \frac{1}{2}\delta^\lambda_\alpha(\Box h)^2 - k_\rho k_\sigma k_\xi\dddot{h}^{\rho\sigma}k_\alpha\dot{h}^\lambda\xi + \\ & + k_\rho k_\sigma\dddot{h}^{\rho\sigma}k^\lambda k_\alpha\dot{h} + k_\alpha\dot{h}^\lambda\xi\Box h_{,\xi} - \Box h^{,\lambda}k_\alpha\dot{h}\Big) \,. \end{aligned} \tag{36}$$

Now remember that

$$\dot{h} = \eta_{\xi\lambda}\dot{h}^{\lambda\xi} \,, \qquad \ddot{h} = \eta_{\xi\lambda}\ddot{h}^{\lambda\xi} \,, \tag{37}$$

and

$$k^\lambda\eta_{\xi\lambda} = k_\xi \,, \tag{38}$$

and then

$$k^\lambda k_\alpha \ddot{h} = k^\lambda k_\alpha \eta_{\xi\lambda} \ddot{h}^{\lambda\xi} = k_\xi k_\alpha \ddot{h}^{\lambda\xi} \,, \tag{39}$$

we can further simplify t^λ_α in the following way

$$\begin{aligned} t^\lambda_\alpha \;&=\; f_0' \left(k^\lambda k_\alpha \dot{h}^{\rho\sigma} \dot{h}_{\rho\sigma} \right) + f_0'' \left(k_\rho k_\sigma \ddot{h}^\rho \sigma k_\xi k_\alpha \ddot{h}^{\lambda\xi} + \right. \\ &\quad - k_\rho k_\sigma \ddot{h}^{\rho\sigma} k^\lambda k_\alpha \ddot{h} - \frac{1}{2} \ddot{h}^{\rho\sigma} \delta^\lambda_\alpha k_\rho k_\sigma \ddot{h}^\rho \sigma - \\ &\quad \left. + k_\rho k_\sigma k_\xi \dddot{h}^{\rho\sigma} k_\alpha \dot{h}^{\lambda\xi} + k_\rho k_\sigma \dddot{h}^{\rho\sigma} k^\lambda k_\alpha \dot{h} \right) \,, \end{aligned} \tag{40}$$

we notice that the sixth and fifth terms of above equation are equal because

$$\begin{aligned} k_\rho k_\sigma k^\lambda k_\alpha \dddot{h}^{\rho\sigma} \dot{h} \;&=\; k_\rho k_\sigma k^\lambda \dddot{h}^{\rho\sigma} \eta_{\xi\lambda} \dot{h}^{\lambda\xi} = \\ &= k_\rho k_\sigma k_\xi \dddot{h}^{\rho\sigma} \dot{h}^{\rho\xi} \,, \end{aligned} \tag{41}$$

the third and fourth are the same, and then t^λ_α reduces to

$$\begin{aligned} t^\lambda_\alpha \;&=\; f_0' \left(k^\lambda k_\alpha \dot{h}^{\rho\sigma} \dot{h}_{\rho\sigma} \right) - \frac{1}{2} f_0'' \left(k_\rho k_\sigma \ddot{h}^{\rho\sigma} \delta^\lambda_\alpha k_\rho k_\sigma \ddot{h}^{\rho\sigma} \right) = \\ &= f_0' \left(k^\lambda k_\alpha \dot{h}^{\rho\sigma} \dot{h}_{\rho\sigma} \right) - \\ &+ \frac{1}{2} f_0'' \left(k_\rho k_\sigma \ddot{h}^{\rho\sigma} \eta^{\lambda\xi} \eta_{\xi\alpha} k_\rho k_\sigma \ddot{h}^{\rho\sigma} \right) \,, \end{aligned} \tag{42}$$

finally the energy momentum tensor assume the following form

$$t^\lambda_\alpha \;=\; \underbrace{f_0' k^\lambda k_\alpha \left(\dot{h}^{\rho\sigma} \dot{h}_{\rho\sigma} \right)}_{GR} - \underbrace{\frac{1}{2} f_0'' \delta^\lambda_\alpha \left(k_\rho k_\sigma \ddot{h}^{\rho\sigma} \right)^2}_{f(R)} \,. \tag{43}$$

To be more precise, the first term, depending on the choice of the constant f_0', is the standard GR term, the second is the $f(R)$ contribution. It is worth noticing that the order of derivative is increased of two degrees consistently to the fact that $f(R)$-gravity is of fourth-order in the metric approach.

Now we could compute the instantaneous $\frac{dE}{dt}$ using the Eq.(43) as a basis. The effect on a binary system is is more evident if we consider the average flux of energy away from the system. Suppose that the $h_{\mu\nu}$ waves can be represented by a discrete spectral representation. The periodicity T will be proportional to the inverse of the difference of the pair of frequency components in the wave. Therefore, we must to evaluate the average of $\frac{dE}{dt}$ over an interval equal to or greater than T [35, 36]. The instantaneous flux of energy through a surface of area $r^2 d\Omega$ in the direction $\hat{x}$

$$\frac{dE}{dt} = r^2 d\Omega \hat{x}^i t^{0i} \,, \tag{44}$$

and the average flux is

$$\left\langle \frac{dE}{dt} \right\rangle = r^2 d\Omega \hat{x}^i \langle t^{0i} \rangle \,, \tag{45}$$

and then Eq. 43 becomes

$$\langle t^\lambda_\alpha \rangle \;=\; \left\langle f_0' k^\lambda k_\alpha \left(\dot{h}^{\rho\sigma} \dot{h}_{\rho\sigma} \right) - \frac{1}{2} f_0'' \delta^\lambda_\alpha \left(k_\rho k_\sigma \ddot{h}^{\rho\sigma} \right)^2 \right\rangle \,. \tag{46}$$

Finally, we re-write $\langle t^\lambda_\alpha \rangle$ in terms of a function $J_{\mu\nu}$ defined to be

$$J_{\mu\nu}(\vec{x}, t) \simeq 4 \int d^3\vec{x}' \frac{T_{\mu\nu}(\vec{x}', t - |\vec{x}' - \vec{x}|)}{|\vec{x}' - \vec{x}|} \,, \tag{47}$$

noting that

$$h_{\mu\nu}(\vec{x}, t) = J_{\mu\nu}(\vec{x}, t) \,, \tag{48}$$

and consequently

$$\dot{h}^{\rho\sigma} \dot{h}_{\rho\sigma} = \dot{J}^{\rho\sigma} \dot{J}_{\rho\sigma} \qquad \ddot{h}^{\rho\sigma} \ddot{h}_{\rho\sigma} = \ddot{J}^{\rho\sigma} \ddot{J}_{\rho\sigma} \tag{49}$$

to give

$$\langle t^\lambda_\alpha \rangle \;=\; \left\langle f_0' k^\lambda k_\alpha \dot{J}^{\rho\sigma} \dot{J}_{\rho\sigma} - \frac{1}{2} f_0'' \delta^\lambda_\alpha \left(k_\rho k_\sigma \right)^2 \ddot{J}^{\rho\sigma} \ddot{J}_{\rho\sigma} \right\rangle \,. \tag{50}$$

IV. MOMENTA AND CONSERVATION LAWS

Let us now analyze the radiation in terms of multipoles, that means to expand $J_{\mu\nu}$ in a Taylor series about $t' = t - r$. That is,

$$\begin{aligned} J^{\mu\nu}(\vec{x}, t) = \frac{4}{r} \Bigg[&\int d^3\vec{x}' T^{\mu\nu}(\vec{x}', t') + \\ &+ \hat{x} \int d^3\vec{x}' \vec{x}' \frac{\partial T^{\mu\nu}(\vec{x}, t)}{\partial t'} + \\ &+ \frac{1}{2} \int d^3\vec{x}' (\hat{x} \cdot \vec{x}')^2 \frac{\partial^2 T^{\mu\nu}(\vec{x}, t)}{\partial t'}^2 \Bigg] \,, \end{aligned} \tag{51}$$

where we have used

$$|\vec{x}' - \vec{x}|^{-1} \simeq \frac{1}{r} \,, \tag{52}$$

and

$$|\vec{x}' - \vec{x}| \simeq r - \hat{x} \cdot \vec{x}' \,, \tag{53}$$

for $r >> |\vec{x}'|$. Let us define the following moments of the mass-energy distribution:

$$M(t) \simeq \int d^3\vec{x}\, T^{00}(\vec{x}, t) \,, \tag{54}$$

$$D^k(t) \simeq \int d^3\vec{x}\, x^k T^{00}(\vec{x}, t) \,, \tag{55}$$

$$Q^{ij}(t) \simeq \int d^3\vec{x}\, x^i x^j T^{00}(\vec{x}, t) \,. \tag{56}$$

The conservation law becomes, in the weak field limit,

$$T^{\mu\nu}{}_{,\nu} = 0\,, \tag{57}$$

and implies the relations [28, 35]

$$\begin{aligned}\int d^3\vec{x}\, T^{jk}(\vec{x},t) \;&=\; \frac{1}{2}\frac{\partial^2}{\partial t^2}\int d^3\vec{x}\, x^j x^k T^{00}(\vec{x},t) = \\ &= \frac{1}{2}\ddot{Q}^{jk}(t)\,,\end{aligned} \tag{58}$$

$$\begin{aligned}\int d^3\vec{x}\, T^{0k}(\vec{x},t) \;&=\; \frac{\partial}{\partial t}\int d^3\vec{x}\, x^k T^{00}(\vec{x},t) = \\ &= \dot{D}^k(t)\,,\end{aligned} \tag{59}$$

$$\begin{aligned}\frac{\partial}{\partial t}\int d^3\vec{x}\, x^k T^{j0}(\vec{x},t) \;&=\; \int d^3\vec{x}\, T^{jk}(\vec{x},t) = \\ &= \frac{1}{2}\ddot{Q}^{jk}(t)\,,\end{aligned} \tag{60}$$

We use Eq.(51) to write J^{00}, J^{0i} and J^{ij} in terms of the momenta Eqs.(54)-(56). First, from Eq.(51) we have

$$\begin{aligned}J^{00}(\vec{x},t) = 4\frac{1}{r}\Bigg[&\int d^3\vec{x}'\, T^{00}(\vec{x}',t') + \\ &+\hat{x}_i\frac{\partial}{\partial t'}\int d^3\vec{x}'\, x'^i T^{00}(\vec{x}',t') + \\ &+\frac{1}{2}\hat{x}_i\hat{x}_j\frac{\partial^2}{\partial t^2}\int d^3\vec{x}'\, x'^i x'^j T^{00}(\vec{x}',t') + ...\Bigg]\,,\end{aligned} \tag{61}$$

For J^{00} , it is easy to obtain

$$J^{00}(\vec{x},t) = \frac{4}{f_0'}\frac{1}{r}\left[M(t') + \hat{x}_i\dot{D}^i(t') + \frac{1}{2}\hat{x}_i\hat{x}_j\ddot{Q}^{ij}(t')\right]. \tag{62}$$

For J^{0i} we need only two terms of the expansion Eq.(51) in order to include terms up to the second momentum

$$\begin{aligned}J^{0i}(\vec{x},t) \;&=\; 4\frac{1}{r}\Bigg[\int d^3\vec{x}'\, T^{0i}(\vec{x}',t') + \\ &\quad +\hat{x}_k\frac{\partial}{\partial t'}\int d^3\vec{x}'\, x'^k T^{0i}(\vec{x}',t')\Bigg]\,,\end{aligned} \tag{63}$$

Eq.(59) and Eq.(60) then give

$$J^{0i}(\vec{x},t) = 4\frac{1}{r}\left[\dot{D}^i(t') + \frac{1}{2}\hat{x}_k\ddot{Q}^{ik}(t')\right]. \tag{64}$$

For J^{ij} only one term of Eq.(51) is required, being

$$J^{ij}(\vec{x},t) = 2\frac{1}{r}\ddot{Q}^{ij}(t')\,. \tag{65}$$

The conservation law also implies that

$$\dot{M} = 0\,, \qquad \ddot{D}^k = 0\,. \tag{66}$$

Furthermore, from Eq.(62), Eq.(64), and Eq.(65) we have

$$\dot{J}^{00} = 2\frac{1}{r}\hat{x}_i\hat{x}_j\dddot{Q}^{ij}\,, \tag{67}$$

$$\dot{J}^{0i} = 2\frac{1}{r}\hat{x}_k\dddot{Q}^{ik}\,, \tag{68}$$

$$\dot{J}^{ij} = 2\frac{1}{r}\dddot{Q}^{ij}\,, \tag{69}$$

and consequently

$$\ddot{J}^{00} = 2\frac{1}{r}\hat{x}_i\hat{x}_j\ddddot{Q}^{ij}\,, \tag{70}$$

$$\ddot{J}^{0i} = 2\frac{1}{r}\hat{x}_k\ddddot{Q}^{ik}\,, \tag{71}$$

$$\ddot{J}^{ij} = 2\frac{1}{r}\ddddot{Q}^{ij}\,, \tag{72}$$

In order to evaluate Eq.(50), we require that

$$\dot{J}^{\rho\sigma}\dot{J}_{\rho\sigma} = \dot{J}^{00}\dot{J}_{00} + 2\dot{J}^{0i}\dot{J}_{0i} + \dot{J}^{ij}\dot{J}_{ij}\,. \tag{73}$$

and

$$\ddot{J}^{\rho\sigma}\ddot{J}_{\rho\sigma} = \ddot{J}^{00}\ddot{J}_{00} + 2\ddot{J}^{0i}\ddot{J}_{0i} + \ddot{J}^{ij}\ddot{J}_{ij}\,. \tag{74}$$

Pluggins Eqs.(67)-(72) into Eq.(73) and (74), we get

$$\begin{aligned}\dot{J}^{\rho\sigma}\dot{J}_{\rho\sigma} \;&=\; \frac{4}{r^2}\Bigg[\left(\hat{x}_i\hat{x}_j\dddot{Q}^{ij}\right)^2 - \\ &\quad -2\left(\hat{x}_k\dddot{Q}^{ik}\right)\left(\hat{x}_j\dddot{Q}^{ij}\right) + \left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right)\Bigg]\,.\end{aligned} \tag{75}$$

In completely analogous way, we find

$$\begin{aligned}\ddot{J}^{\rho\sigma}\ddot{J}_{\rho\sigma} \;&=\; \frac{4}{r^2}\Bigg[\left(\hat{x}_i\hat{x}_j\ddddot{Q}^{ij}\right)^2 - \\ &\quad -2\left(\hat{x}_k\ddddot{Q}^{ik}\right)\left(\hat{x}_j\ddddot{Q}^{ij}\right) + \left(\ddddot{Q}^{ij}\ddddot{Q}_{ij}\right)\Bigg]\,.\end{aligned} \tag{76}$$

When Eq.(75) and Eq.(76) are put into Eq.(50), we find

$$\begin{aligned}\langle t^{\lambda}_{\alpha}\rangle \;&=\; \Bigg\langle f_0' k^{\lambda}k_{\alpha}\frac{4}{r^2}\Bigg[\left(\hat{x}_i\hat{x}_j\dddot{Q}^{ij}\right)^2 - 2\left(\hat{x}_k\dddot{Q}^{ik}\right)\left(\hat{x}_j\dddot{Q}^{ij}\right) + \\ &\quad +\left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right)\Bigg] - f_0''\delta^{\lambda}_{\alpha}\left(k_{\rho}k_{\sigma}\right)^2\frac{2}{r^2}\Bigg[\left(\hat{x}_i\hat{x}_j\ddddot{Q}^{ij}\right)^2 + \\ &\quad -2\left(\hat{x}_k\ddddot{Q}^{ik}\right)\left(\hat{x}_j\ddddot{Q}^{ij}\right) + \left(\ddddot{Q}^{ij}\ddddot{Q}_{ij}\right)\Bigg]\Bigg\rangle\,.\end{aligned} \tag{77}$$

Using the result in Eq.(45) and integrating over all directions in order to compute the total average flux of energy due to the tensor wave,

$$\left\langle \frac{dE}{dt} \right\rangle_{(total)} = r^2 \int d\Omega \hat{x}^i \langle t^{0i} \rangle \,. \tag{78}$$

Note that

$$\hat{x}^\alpha \langle t^{0i} \rangle = \hat{x}^i k^0 k^i [...] = \hat{x}^i (-1)(-\hat{x}^i)[...] = [...] \,, \tag{79}$$

which simplify the evaluation of Eq.(78). Integration over all direction is accomplished readily with the help of

$$\int d\Omega \hat{x}^i \hat{x}^j = \frac{4\pi}{3} \delta_{ij} \,, \tag{80}$$

and

$$\int d\Omega \hat{x}^i \hat{x}^j \hat{x}^l \hat{x}^m = \frac{4\pi}{15} \left(\delta_{ij}\delta_{lm} - \delta_{il}\delta_{jm} \right) . \tag{81}$$

The result is:

$$\underbrace{\left\langle \frac{dE}{dt} \right\rangle}_{(total)} = \frac{G}{60} \left\langle \underbrace{f_0' \left(\dddot{Q}^{ij} \dddot{Q}_{ij} \right)}_{GR} - \underbrace{f_0'' \left(\dddot{Q}^{ij} \dddot{Q}_{ij} \right)}_{f(R)} \right\rangle . \tag{82}$$

Precisely, for $f_0'' \to 0$ and $f_0' \to \frac{4}{3}$, Eq.(82) becomes

$$\underbrace{\left\langle \frac{dE}{dt} \right\rangle}_{(GR)} = \frac{G}{45} \left\langle \dddot{Q}^{ij} \dddot{Q}_{ij} \right\rangle . \tag{83}$$

which is which is the well-known result of GR [28, 36]. See also [46] for the recovering of the correct GR-limit.

An important remark is necessary at this point. Eq.(82) can be written as

$$\underbrace{\left\langle \frac{dE}{dt} \right\rangle}_{(total)} = \frac{G f_0'}{60} \left\langle \left(\dddot{Q}^{ij} \dddot{Q}_{ij} \right) - \frac{1}{m^2} \left(\dddot{Q}^{ij} \dddot{Q}_{ij} \right) \right\rangle . \tag{84}$$

where the massive mode contribution is evident. This means that this further term affects both the total energy release as well as the waveform. This could represent a further signature to investigate such theories in the GW strong-field regime.

V. APPLICATION TO THE BINARY SYSTEMS: THE PSR 1913+16 CASE

Observations coming from PSR 1913+16 can be used to fix bounds on $f(R)$ parameters. This could be consider a new test to retain or exclude such theories beside the classical Solar System experiments adopted for GR [28]. For a binary system, we have to assume that the motion is Keplerian in the first approximation and we can average over orbital periods. Given a point mass m, $Q^{ij}(t)$ is

$$\begin{aligned} Q^{ij}(t) &\equiv \int d^3\mathbf{x}\, x^i x^j T^{00}(\mathbf{x},t) \equiv \\ &\int\int\int d\underline{x}^1 d\underline{x}^2 d\underline{x}^3\, m\, \underline{x}^i \underline{x}^j \times \\ &\times \delta\left(\underline{x}^1 - x^1(t)\right) \delta\left(\underline{x}^2 - x^2(t)\right) \delta\left(\underline{x}^3 - x^3(t)\right) , \end{aligned} \tag{85}$$

where $\underline{x}$ is the integration variable and x is the position of the mass [28]. We define m as the pulsar mass, M the mass of the companion star, and $\mu = \frac{GM^3}{(M+m)^2}$ the reduced mass. This last definition will be used to account for the fact that m can be small with respect to M. Since the orbit is Keplerian, we can choose $x^3 = 0$ being a planar motion. Then Eq.(85) reduces to

$$Q^{ij}(t) = 0 \,, \qquad \text{for } i \text{ and/or } j = 3 \,, \tag{86}$$

$$Q^{11}(t) = m(x^1(t))^2 \,, \quad Q^{22}(t) = m(x^2(t))^2 \,, \tag{87}$$

$$Q^{12}(t) = Q^{12}(t) = m\left[x^1(t))(x^2(t)\right] , \tag{88}$$

where the position in the orbital plane is a function of time [28]. We are going to work in a parametric representation of the motion [36, 39–41] and then let us recast the variables as

$$\begin{aligned} r &= a\,(1 - \epsilon\cos\mathcal{E}) \,, \quad t = \sqrt{\frac{a^3}{\mu}}(\mathcal{E} - \epsilon\sin\mathcal{E}) \,, \\ x^1(\mathcal{E}) &= a\,(\cos\mathcal{E} - \epsilon) \,, \quad x^2(\mathcal{E}) = a\left(1-\epsilon^2\right)^{\frac{1}{2}}\sin\mathcal{E} \,, \end{aligned} \tag{89}$$

where r is the orbital radius, a, the semi-major axis of the orbit, ϵ, the eccentricity, $\mathcal{E}$. the eccentricity anomaly. Over the whole orbit, $\mathcal{E}$ ranges from 0 to 2π. We use $\mathcal{E}$, rather than t, to locate the body in its orbit, and therefore we have to integrate over $d\mathcal{E}$

$$\langle f \rangle \equiv \frac{1}{T} \int_0^T dt f(t) \,. \tag{90}$$

For a Keplerian orbit T has the value

$$T = 2\pi \sqrt{\frac{a^3}{\mu}} \,. \tag{91}$$

Therefore, if $f(t) = g(\mathcal{E}(t))$, we may write Eq.(90) as

$$\langle f(t) \rangle = \frac{1}{2\pi} \int_0^\pi g(\mathcal{E})(1 - \epsilon\cos\mathcal{E}) d\mathcal{E} \,. \tag{92}$$

$\dddot{Q}^{ij}$ can be expressed in terms of the eccentric anomaly and then Eq.(92) can be used to compute the time average over an orbital period[6]. We find that time derivative can be recast as

$$\frac{d}{dt} = \frac{d\mathcal{E}}{dt}\frac{d}{d\mathcal{E}} = \frac{2\pi}{T}(1-\epsilon\cos\mathcal{E})^{-1}\frac{d}{d\mathcal{E}} . \tag{93}$$

From Eqs.(86)-(88), we can write

$$\dddot{Q}^{ij}\dddot{Q}_{ij} = \left(\dddot{Q}^{11}\right)^2 + 2\left(\dddot{Q}^{12}\right)^2 + \left(\dddot{Q}^{22}\right)^2 . \tag{94}$$

Let us consider the various orders of derivation. First, from Eq.(87) and Eq.(89) we have

$$Q^{11}(\mathcal{E}) = ma^2\left(\cos\mathcal{E}-\epsilon\right)^2 . \tag{95}$$

Using Eq.(93) to compute the derivatives, we find

$$\dot{Q}^{11}(\mathcal{E}) = -2ma^2\left(\frac{2\pi}{T}\right)\frac{\sin\mathcal{E}\left(\cos\mathcal{E}-\epsilon\right)}{1-\epsilon\cos\mathcal{E}} , \tag{96}$$

$$\begin{aligned}\ddot{Q}^{11}(\mathcal{E}) &= -2ma^2\left(\frac{2\pi}{T}\right)^2(1-\epsilon\cos\mathcal{E})^{-3}\times\\ &\times\left(2\cos^2\mathcal{E}-\epsilon\cos\mathcal{E}-\epsilon\cos^3\mathcal{E}+\epsilon^2-1\right) ,\end{aligned}$$

and

$$\begin{aligned}\dddot{Q}^{11}(\mathcal{E}) &= -2ma^2\left(\frac{2\pi}{T}\right)^3(1-\epsilon\cos\mathcal{E})^{-5}\sin\mathcal{E}\times\\ &\times\left(\epsilon\cos^2\mathcal{E}+2\epsilon^2\cos\mathcal{E}-4\cos\mathcal{E}\epsilon^3+4\epsilon\right) ,\end{aligned} \tag{97}$$

finally

$$\begin{aligned}\ddddot{Q}^{11}(\mathcal{E}) &= -2ma^2\left(\frac{2\pi}{T}\right)^4(1-\epsilon\cos\mathcal{E})^{-7}\times\\ &\times\left[\left(16\epsilon^3+8\epsilon^2+4\right)\cos 2\mathcal{E}+(8\epsilon^2-3\epsilon)\cos\mathcal{E}+\right.\\ &\left.+3\epsilon(\cos 3\mathcal{E}-4\epsilon(2\epsilon+1))\right] .\end{aligned} \tag{98}$$

Likewise, from Eq.(87) and Eq.(89), we have

$$Q^{22}\mathcal{E} = ma^2\left(1-\epsilon^2\right)\sin^2\mathcal{E} , \tag{99}$$

which leads to the derivatives

$$\dot{Q}^{22}(\mathcal{E}) = 2ma^2\left(\frac{2\pi}{T}\right)\left(1-\epsilon^2\right)\frac{\sin\mathcal{E}\cos\mathcal{E}}{(1-\epsilon\cos\mathcal{E})} ,$$

$$\begin{aligned}\ddot{Q}^{22}(\mathcal{E}) &= 2ma^2\left(\frac{2\pi}{T}\right)^2\frac{\left(1-\epsilon^2\right)}{(1-\epsilon\cos\mathcal{E})^3}\times\\ &\times\left(\cos^2\mathcal{E}-\sin^2\mathcal{E}-\epsilon\cos^3\mathcal{E}\right) ,\end{aligned}$$

$$\begin{aligned}\dddot{Q}^{22}(\mathcal{E}) &= 2ma^2\left(\frac{2\pi}{T}\right)^3\frac{\left(1-\epsilon^2\right)}{(1-\epsilon\cos\mathcal{E})^5}\times\\ &\times\sin\mathcal{E}\left(3\epsilon-4\cos\mathcal{E}+\epsilon^2\mathcal{E}\right) .\end{aligned} \tag{100}$$

$$\begin{aligned}\ddddot{Q}^{22}(\mathcal{E}) &= 2ma^2\left(\frac{2\pi}{T}\right)^4\frac{\left(1-\epsilon^2\right)}{(1-\epsilon\cos\mathcal{E})^7}\times\\ &\times\left[\left(22\epsilon^2-16\right)\cos 2\mathcal{E}+41\epsilon\cos\mathcal{E}+\right.\\ &\left.+\left(\epsilon^2(\cos 4\mathcal{E}-39)-9\epsilon\cos 3\mathcal{E}\right)\right]\end{aligned} \tag{101}$$

Finally, from Eq.(88) and Eq.(89) we have

$$Q^{12} = Q^{21} = ma^2(1-\epsilon)^{\frac{1}{2}}\sin\mathcal{E}(\cos\mathcal{E}-\epsilon) , \tag{102}$$

whose derivatives are

$$\begin{aligned}\dot{Q}^{12} &= ma^2\left(\frac{2\pi}{T}\right)(1-\epsilon)^{\frac{1}{2}}(1-\epsilon\cos\mathcal{E})^{-1}\times\\ &\times(2\cos^2\mathcal{E}-\epsilon\cos\mathcal{E}-1) ,\end{aligned}$$

$$\begin{aligned}\ddot{Q}^{12} &= ma^2\left(\frac{2\pi}{T}\right)^2(1-\epsilon)^{\frac{1}{2}}(1-\epsilon\cos\mathcal{E})^{-3}\sin\mathcal{E}\times\\ &\times\left(2\epsilon\cos^2\mathcal{E}-4\cos\mathcal{E}+2\epsilon\right) ,\end{aligned}$$

$$\begin{aligned}\dddot{Q}^{12} &= ma^2\left(\frac{2\pi}{T}\right)^3(1-\epsilon)^{\frac{1}{2}}(1-\epsilon\cos\mathcal{E})^{-5}\times\\ &\times\left(\epsilon^2\cos^2\mathcal{E}+3\epsilon\cos\mathcal{E}+\epsilon\cos^3\mathcal{E}-\right.\\ &\left.-3\epsilon^2-4\cos^2\mathcal{E}+2\right) ,\end{aligned} \tag{103}$$

$$\begin{aligned}\ddddot{Q}^{12} &= ma^2\left(\frac{2\pi}{T}\right)^4(1-\epsilon)^{\frac{1}{2}}(1-\epsilon\cos\mathcal{E})^{-7}\times\\ &\times\sin\mathcal{E}\left[\left(15\epsilon^2+4\right)\cos\mathcal{E}+\left(3\epsilon^3+6\epsilon\right)\cos 2\mathcal{E}-\right.\\ &\left.+27\epsilon^3+\epsilon^2\cos 3\mathcal{E}+18\epsilon\right]\end{aligned} \tag{104}$$

When results from Eqs.(97), (100), and (103), together with Eq.(91) for T, are used in Eq.(94), one finds

$$\dddot{Q}^{ij}\dddot{Q}_{ij} = 4m^2\frac{\mu^3}{a^5}\frac{\left[8\left(1-\epsilon\right)+\epsilon^2\sin^2\mathcal{E}\right]}{(1-\epsilon\cos\mathcal{E})^6} . \tag{105}$$

[6] Note that we can rise/lower space indices without regard for sign changes because $\eta^{ij} = \delta^{ij}$.

and

$$\dddot{Q}^{ij}\dddot{Q}_{ij} = 2m^2\frac{\mu^4}{a^8}\frac{1}{(1-\epsilon\cos\mathcal{E})^{14}}\times \\ \times 2(\epsilon^2-1)^2\left[41\epsilon\cos\mathcal{E}-9-39\epsilon^2+ \right. \\ \left. +(22\epsilon^2-16)\cos 2\mathcal{E}+\epsilon\cos 4\mathcal{E}\right]^2 + \\ +(1-\epsilon)\left[18\epsilon-27\epsilon^3+(4+15\epsilon^2)\cos\mathcal{E}+ \right. \\ \left. +(6\epsilon+3\epsilon^3)\cos 2\mathcal{E}+\epsilon^2\cos 3\mathcal{E}\right]^2\sin^2\mathcal{E} + \\ +2\left[(8\epsilon^2-3\epsilon)\cos\mathcal{E}+(4+8\epsilon^2+16\epsilon^3)\cos\mathcal{E}+ \right. \\ \left. 2\epsilon\cos 3\mathcal{E}-16\epsilon^3-8\epsilon^2\right]^2 \tag{106}$$

Substituting Eq.(105) into Eq.(92) and averaging, we have

$$\left\langle \ddot{Q}^{ij}\ddot{Q}_{ij}\right\rangle = \frac{4}{\pi}m^2\frac{\mu^3}{a^5}\int_0^\pi \frac{8\,(1-\epsilon)+\epsilon^2\sin^2\mathcal{E}}{(1-\epsilon\cos\mathcal{E})^5}d\mathcal{E}\,. \tag{107}$$

also for Eq. (106) we obtain

$$\left\langle \dddot{Q}^{ij}\dddot{Q}_{ij}\right\rangle = \frac{m^2}{\pi}\frac{\mu^4}{a^8}\int_0^\pi \frac{1}{(1-\epsilon\cos\mathcal{E})^{13}}\times \\ \times 2(\epsilon^2-1)^2\left[41\epsilon\cos\mathcal{E}-9-39\epsilon^2+ \right. \\ \left. +(22\epsilon^2-16)\cos 2\mathcal{E}+\epsilon\cos 4\mathcal{E}\right]^2 + \\ +(1-\epsilon)\left[18\epsilon-27\epsilon^3+(4+15\epsilon^2)\cos\mathcal{E}+ \right. \\ \left. +(6\epsilon+3\epsilon^3)\cos 2\mathcal{E}+\epsilon^2\cos 3\mathcal{E}\right]^2\sin^2\mathcal{E} + \\ +2\left[(8\epsilon^2-3\epsilon)\cos\mathcal{E}+(4+8\epsilon^2+16\epsilon^3)\cos\mathcal{E}+ \right. \\ \left. 2\epsilon\cos 3\mathcal{E}-16\epsilon^3-8\epsilon^2\right]^2 d\mathcal{E} \tag{108}$$

The first term of Eq. (107) is evaluates using

$$\int_0^\pi \frac{d\mathcal{E}}{(1-\epsilon\cos\mathcal{E})^5} = \frac{\pi}{(1-\epsilon^2)^{\frac{5}{2}}}P_4\left(\frac{1}{\sqrt{1-\epsilon^2}}\right)\,,$$

where $P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)$.

$$\int_0^\pi \frac{d\mathcal{E}}{(1-\epsilon\cos\mathcal{E})^5} = \frac{\pi}{8}\frac{3\epsilon^4+24\epsilon^2+8}{(1-\epsilon^2)^{\frac{9}{2}}}\,. \tag{109}$$

The complete evaluation of Eq. (107) is

$$\left\langle \ddot{Q}^{ij}\ddot{Q}_{ij}\right\rangle = \frac{1}{2}m^2\frac{\mu^3}{a^5}\frac{25\epsilon^4+196\epsilon^2+64}{(1-\epsilon^2)^{\frac{7}{2}}}\,. \tag{110}$$

and for the Eq. (108) we do not have an analytical solution of the integral but, only a numerical result that will be insert in the following equations.

The above results apply for the motion of a body of mass m in a Keplerian orbit about a second body of mass M. Therefore, we can evaluate the overall loss rate due to the motion of both bodies. Let the subscript 1 denote the position of the pulsar m and 2 that of the companion M and, as above, let the coordinate origin be at the barycenter [42, 43]. This condition gives

$$mx_1^i + Mx_2^i = 0\,, \tag{111}$$

and then

$$x_2^i = -\frac{m}{M}x_1^i\,. \tag{112}$$

The averall momentum Q^{ij} for the system consisting of both m and M is

$$Q^{ij} = mx_1^ix_1^j + Mx_2^ix_2^j = \frac{m}{M}(m+M)x_1^ix_1^j\,, \tag{113}$$

where we have used Eq.(112) to express the momentum in terms of the motion of m. Averaging for the binary system, we obtain

$$\left\langle \dddot{Q}^{ij}\dddot{Q}_{ij}\right\rangle = \frac{G^3m^2M^7}{2a^5(m+M)^4}\frac{25\epsilon^4+196\epsilon^2+64}{(1-\epsilon^2)^{\frac{7}{2}}}\,. \tag{114}$$

We do not measure $\frac{d\mathcal{E}}{dt}$ directly but the change of orbital period T induced by $\frac{d\mathcal{E}}{dt}$. To this end, we remember that the semi-major axis of the orbit is [36, 39, 42–44]

$$a' = \frac{m+M}{M}a\,, \tag{115}$$

where it has to be recalled that a is the semi-major axis of the pulsar orbit. The total energy E of a Keplerian binary system is then

$$E = -\frac{GmM}{2a'} = -\frac{GmM^2}{2a(m+M)}\,, \tag{116}$$

from which

$$a = -\frac{GmM^2}{2(m+M)}\frac{1}{E}\,. \tag{117}$$

The orbital period T can be related to the energy E by combining Eq.(91) and Eq.(117). The result is

$$T = -\pi GE^{\frac{3}{2}}\left(\frac{m^3M^3}{2(m+M)}\right)^{\frac{1}{2}}\,. \tag{118}$$

By taking the time derivative and Eq.(116) to restore the parameter a, we find that

$$\frac{dT}{dt} = \dot{T} = 6\pi\frac{(m+M)^2}{m}\sqrt{\frac{a^5}{G^3M^7}}\frac{dE}{dt}\,. \tag{119}$$

Let us now use the published numerical values for the specific example of PSR 1913+16 to numerically evaluate the above equations . The results will be included into Eq.(82) to evaluate $\left\langle \frac{dE}{dt} \right\rangle$ from which $\frac{dT}{dt}$ can be estimated using Eq.(119). We use the values from Taylor et al. [1, 2] for PSR 1913+16 reported in Table I.

PSR 1913+16	chacteristic features
pulsar mass	$m = 1.39 M_\odot$
companion mass	$M = 1.44 M_\odot$
inclination angle	$\sin i = 0.81$
orbit semimajor axis	$a = 8.67 \times 10^{10} cm$
eccentricity	$\epsilon = 0.617155$
gravitational constant	$G = 6.67 \times 10^{-8} dyn\, cm^2\, g^{-2}$
speed of light	$c = 2.99 \times 10^{10} cm\, sec^{-1}$

Table I: Values from Taylor et al. for PSR 1913+16 [1, 2].

First we find, from Eq.(119), that for PSR 1913+16

$$\dot{T} = 2.21 \times 10^{-44} \left(\frac{sec^3}{g\, cm^2} \right) \frac{dE}{dt} . \tag{120}$$

$$\left\langle \left(\dddot{Q}^{ij} \right)^2 \right\rangle = 2.78 \times 10^{92} \left(\frac{g\, cm^2}{sec^3} \right)^2 , \tag{121}$$

$$\left\langle \left(\ddddot{Q}^{ij} \right)^2 \right\rangle = 1.29 \times 10^{96} \left(\frac{g\, cm^2}{sec^3} \right)^2 \tag{122}$$

Then, the averaged total radiation rate for the binary system in $f(R)$- gravity is found from Eq.(82), that is [7]

$$\left\langle \frac{dE}{dt} \right\rangle = \left[1.32 \times 10^{31} f'_0 - 6.10 \times 10^{34} f''_0 \right] \left(\frac{g\, cm^2}{sec^3} \right) . \tag{123}$$

Using this value in Eq.(120), we find

$$\dot{T}_{f(R)} = 2.92 \times 10^{-13} f'_0 - 1.34 \times 10^{-9} f''_0 \left(\frac{sec}{sec} \right) , \tag{124}$$

as we can see from the above equation the orbital period depends strongly on the choice theory. Now, given the value of f'_0, (*i.e.* $f'_0 = \frac{4}{3}$), we determine the value of f''_0 that falls within the limits observed by Hulse and Taylor [1, 2]. We remember that they predicted an upper and lower limit in the observation of the orbital period that is about $3.8 \times 10^{-12} \leq \dot{T} \leq 2.6 \times 10^{-12}$ and the limit for GR is

$$\dot{T}_{GR} \simeq 3.36 \times 10^{-12} \left(\frac{sec}{sec} \right) . \tag{125}$$

We immediately recover GR limit putting $f'_0 = \frac{4}{3}$ and $f''_0 = 0$. In Fig. 1 is shown a plot of $\dot{T}$ from (124) for PSR 1913+16 as a function of f''_0 parameter. In

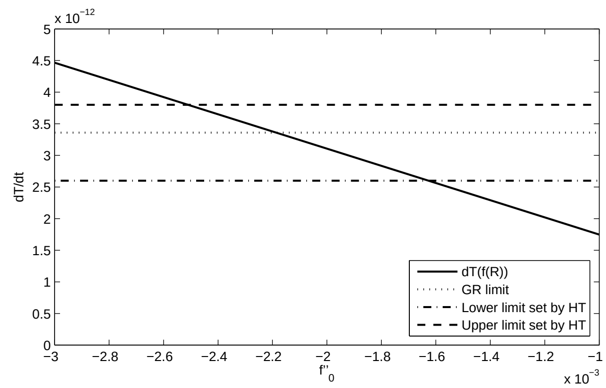

Figure 1: Orbital decay rate for PSR 1913+16 in $f(R)$-gravity. Upper limit set by Taylor et al. in dashed line. GR limit 3.36×10^{-12} in dotted line and the lower limit set by Taylor et al. in dashdot line. While in solid line is plotted $\dot{T}_{f(R)}$.

Fig. 1, the observational limits on $\dot{T}$ are indicated together with the GR limiting value (125). The range $-2.63 \times 10^{-3} \leq f''_0 \leq -2.25 \times 10^{-3}$ well fits with these observational limits [1, 2]. In other words, we can conclude that $f(R)$-gravity is not excluded by the Hulse and Taylor observations on binary pulsar. On the other hand, such observations contribute to fix the range of viability of such theories.

VI. CONCLUDING REMARKS

In this paper, we developed the post-Minkowskian limit of analytic $f(R)$-gravity models in the Jordan frame to calculate the gravitational radiation emitted by a binary system. One of the results is that the quadrupole-radiation, in $f(R)$-gravity and in GR, occurs independently of the detailed internal structure of the stellar bodies. It depends on the masses of the two bodies, on the orbital parameters and on the details of the gravitational theory. Further massive modes emerge and they are directly related to the analytic parameters of $f(R)$-gravity, that is the coefficients f'_0 and f''_0 of the Taylor expansion. This fact is relevant since it does not depend on specific $f(R)$-models but it is a general feature.

As a consequence, the theoretical quadrupole radiation rate, calculated according to the theory, can be confronted to binary system observations to fix the parameters of the theory. Specifically, the radiation rate is a

[7] Note that for dimensional reasons we have to restore the factor of c^5 to the denominator of $\frac{dE}{dt}$.

function of f_0' and f_0''. As we can see from Fig. 1 or, equivalently from Eq. (124), the predicted range of the time derivative of the orbital period for PSR 1913+16 is compatible with the observational uncertainty established by Hulse and Taylor. [45]. This means that observations can fix the parameters of the theory. These results pose interesting problems related to the strict validity of GR. It seems that it works very well at local scales (Solar System) where effects of further gravitational degrees of freedom cannot be detected. As soon as one is investigating larger scales, as those of galaxies, clusters of galaxies, etc., further corrections can be introduced in order to explain both astrophysical large-scale dynamics [46] and cosmic evolution [47, 48]. Alternatively, huge amounts of dark matter and dark energy have to be invoked to explain the phenomenology, but, up today there are no final evidences for these new constituents at fundamental level. What we have shown is that the Hulse and Taylor experiment, beside confirming GR, does not exclude Extended Theories of Gravity [20] including GR as a particular case.

Furthermore, the fact that, up to now, only massless gravitational waves have been investigated could be a shortcoming preventing the possibility to find out other forms of gravitational waves. Tests in this sense could come out, for example, from the stochastic background of gravitational waves where massive modes could play a crucial role in the cosmic background spectrume [49, 50].

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